2013 数理 問3.

[1]

$$G_X(s) = E[s^x] = \sum {}_nC_x (sp)^x (1-p)^{n-x}$$

$$= (sp-1-p)^n.$$

$$G'_X(s) = n(sp-1-p)^{n-1} \cdot p \Big|_{s=1} np = E[X]$$

$$G''_X(s) = np(n-1)(sp-1-p)^{n-2} \cdot p \Big|_{s=1} n(n-1)p^2 = E[X(X-1)]$$

$$V[X] = E[X^2] - (E[X])^2 = E[X(X-1)] + E[X] - \{E[X]\}^2$$

$$= n^2p^2 - np^2 + np - n^2p^2 = np(1-p)」$$

[2]

$$l(p) = \log {}_nC_x + x \log p + (n-x)\log(1-p).$$

$$l'(p) = \frac{x}{p} - \frac{n-x}{1-p} = 0 \iff \hat{p} = \frac{x}{n}$$

$$E[\hat{p}] = E\left[\frac{X}{n}\right] = \frac{1}{n}E[X] = \frac{1}{n}np = p」$$

[3]

$$V[\hat{p}] = V\left[\frac{X}{n}\right] = \frac{1}{n^2}np(1-p) = \frac{p(1-p)}{n} = w_n(p)$$

$$\hat{w}_n = w_n(\hat{p}) = \frac{1}{n} \cdot \frac{X}{n}\left(1-\frac{X}{n}\right)$$

$$E[\hat{w}_n] = E\left[\frac{1}{n} \cdot \frac{X}{n}\left(1-\frac{X}{n}\right)\right] = \frac{1}{n^3}E\left[nX - X^2\right] = \frac{1}{n^3}\left\{E[nX] - E[X^2]\right\}.$$

$$= \frac{1}{n^3}\left\{n^2p - (n^2p^2 + np(1-p))\right\} = \frac{1}{n^3}(n^2p - n^2p^2 - np + np^2) = \frac{p}{n^2}(n - np - 1 + p)$$

$$= \frac{p}{n^2}\left\{n(1-p) - (1-p)\right\} = \frac{p}{n^2}(1-p)(n-1) = \frac{p(1-p)}{n} \cdot \frac{n-1}{n}.$$

$$C(n) = \frac{n}{n-1}$$

[4] $\frac{n}{n-1} > 1$ より.

$\hat{p} \pm 1.96\sqrt{\hat{w}_n}$ よりも $\hat{p} \pm 1.96\sqrt{\tilde{w}_n}$ は区間が広くなる.